Some finite index subgroups of the braid group B_3

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The substantial content of my paper is purely algebraic, however its motivation is topological. It was known, that B_3 is a fundamental group of a three-manifold. Namely it is a fundamental group of the complement of trefoil knot. Questions I answer in my manuscript are significant to Heegaard splittings.

The aforementioned group B_3 is defined as the group of all braids of three strands. Important conditions is that all strands must go strictly down at any given position. In other worlds any strand cannot change its vertical direction, but it can freely move along other two axis while going down.

I start my work by giving the presentation $\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ of B_3.
I study subgroups of B_3 with finite index. My purpose is to find what form those subgroups can have, and what they cannot. The most important result in this paper follows from considering homomorphism from braid group B_3 to the symmetry group of a set with tree elements S_3. This homomorphism maps every braid to the permutation of strands it gives. Group S_3 has a presentation $\langle g_1, g_2 \mid (g_1g_2)^3 = g_1^2 = g_2^2 = e \rangle$, which we use to determine what kernel looks like. This kernel is a subgroup of B_3 with finite index. To be precise, its index is equal to the number of elements in S_3, that is 6.

Given presentation mentioned above we can conclude that kernel is a minimal normal subgroup containing a group generated by elements ${\sigma_1}^2$,σ_2^2,$(\sigma_1\sigma_2)^3$. We give a proof that a group generated by those elements is itself a normal subgroup of B_3 and it is isomorphic to $\mathbb{Z} \times F_2$, where $\langle (\sigma_1\sigma_2)^3 \rangle \simeq \mathbb{Z}$, and $\langle \sigma_1^2, \sigma_2^2 \rangle \simeq F_2$.

While studying the subgroup $\langle \sigma_1^2, \sigma_2^2 \rangle$, it is worth to consider matrices, more precisely the group $SL_2(\mathbb{Z})$, on which we can map B_3. Matrices are useful in this case as they act on a plane in a natural way. This helps us apply Ping-Pong Lemma, which would be hard to use directly on B_3. It turns out, that the homomorphism from B_3 to S_3 mentioned earlier can be constructed with the use of homomorphism from B_3 to $SL_2(\mathbb{Z})$.
It is a consequence of the fact, that $SL_2(\mathbb{Z}_2) \simeq S_3$.

After showing that kernel of homomorphism from B_3 to $SL_2(\mathbb{Z})$ is isomorphic to the group $\mathbb{Z} \times F_2$ I present a family of subgroups B_3 with finite index of the form $\mathbb{Z} \times F_n$ for each $n \geq 2$. I also show that the group B_3 is not virtually of the form F_n for any $n \in \mathbb{Z}^+$.