Number, Sum and Product of Divisors of Positive Integer

Nikita Arskyi (Ukraine)

Summary

The following functions are defined for any positive integer n

$$\tau(n)=\sum_{d|n}1\,,\sigma(n)=\sum_{d|n}d\,,\sigma_m(n)=\sum_{d|n}d^m,\pi(n)=\prod_{d|n}d$$

as the number, sum, sum of the m-th ($m\in\mathbb{Z}^+$) powers, and product of the divisors of n, respectively.

The aim of the research is to widen the range of properties of these functions and to prove their theoretical and practical usefulness. Methods of the research: gathering, analyzing, and systematizing known facts, searching for new patterns, formulating and proving the propositions and the properties, creating and solving the problems.

As a result, the properties of the explored functions were supplemented with the following propositions: the theorems about the values of the functions for the product of two positive integers were generalized; the number, sum, and product of the divisors of a positive integer that are divisible by a certain integer or are perfect powers, and all the common divisors of two positive integers were derived. The formulae for evaluating the following expressions were obtained: $\tau(\mathrm{gcd}(s;t))\tau\big(\mathrm{lcm}(s;t)\big)$, $\sigma_m(\mathrm{gcd}(s;t))\sigma_m\big(\mathrm{lcm}(s;t)\big)$ and $\pi(\mathrm{gcd}(s;t))^{\tau(\mathrm{lcm}(s;t))}\pi\big(\mathrm{lcm}(s;t)\big)^{\tau(\mathrm{gcd}(s;t))}$ ($\mathrm{gcd}(s;t)$ and $\mathrm{lcm}(s;t)$ denote the greatest common divisor and the least common multiple of the positive integers s and t, respectively).

Some of the mean values of the divisors of a positive integer and their properties were researched. The formulae for evaluating the sums of all possible products each of which contains 2 or 3 divisors of a positive integer were derived together with the recurrence formula

$$s_{k+1,m}(n)=\frac{1}{k+1}\left((-1)^k\sigma_{(k+1)m}(n)+\sum_{l=0}^{k-1}(-1)^l s_{k-l,m}(n)\sigma_{(l+1)m}(n)\right),$$

where $d_1,d_2,\dots,d_{\tau(n)}$ are all the divisors of n in increasing order, $s_{k,m}(n)=\sum_{1\le a_1<a_2<\cdots<a_k\le\tau(n)}d_{a_1}^m d_{a_2}^m\cdot\ldots\cdot d_{a_k}^m$, $k\in\mathbb{Z}^+, k\le\tau(n)$.

During the research, it was discovered that the issue of divisors was raised a long time ago, but the interest in this topic has not faded yet. In the Mathematics community, much attention is paid to the divisors of a positive integer, and there are numerous sources in which they are considered. Nevertheless, in the majority of them, the number, sum, and product of divisors functions are not used as a tool for the search for answers. This causes a negative impact on the productivity of the corresponding calculations and does not allow one to look at the divisors from a different angle and to widen the range of their properties. Thus, the obtained results are important from both theoretical and practical point of view.